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RESONANCE OSCILLATIONS OF BEAM EQUATIONS

Dedicated to Professor Takaŝi Kusano on his sixtieth birthday

Norio YOSHIDA

1. Introduction

In this paper we consider the beam equation

$$(*) \qquad \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u}{\partial x^2} + \sigma \frac{\partial u}{\partial t} + \rho \frac{\partial^5 u}{\partial x^4 \partial t} = f(x,t),$$

$$(x,t) \in (0,L) \times \mathbf{R}_+ ,$$

where $\mathbf{R}_+ = (0,\infty)$ and L is a positive number. We assume throughout this paper that:

(A_1) α is a positive constant and β, σ, ρ are constants;

(A_2) $f(x,t)$ is a real-valued continuous function on $\overline{(0,L) \times \mathbf{R}_+}$.

The boundary condition to be considered is the following:

(B) $u(0,t) = g(t),\ \dfrac{\partial^2 u}{\partial x^2}(0,t) = \tilde{g}(t),\ u(L,t) = h(t),\ \dfrac{\partial^2 u}{\partial x^2}(L,t) = \tilde{h}(t),$

$t > 0,$

where $g(t)$, $\tilde{g}(t)$, $h(t)$, $\tilde{h}(t)$ are functions of class $C^1(\mathbf{R}_+)$.

Forced oscillations of the beam equation $(*)$ were investigated by the author [6]. Our purpose is to study the resonance oscillation of the beam equation $(*)$. Here the resonance oscillation means that the amplitude of the solution of the boundary value problem $(*)$, (B) blows up if the forcing term $f(x,t)$ and the functions appearing in (B) are periodic functions of t with period $2\pi/p$ and the frequency $p/2\pi$ approaches some frequency which is characterized by $(*)$ (cf. Timoshenko, Young and Weaver [5]). Resonance oscillations of wave equations were treated by the author [7]. However, to the author's knowledge, there appears to be no known resonance theorems for the beam equation $(*)$.

The existence of solutions to initial-boundary value problems for $(*)$ was discussed by several authors; see, for example, [1–4]. By a "solution" of the boundary value problem $(*)$, (B) we mean a classical solution of the problem. Associated with every solution u of $(*)$, we define

(1) $$U(t) = \frac{1}{L}\int_0^L u(x,t)\,\sin\frac{\pi}{L}x\,dx, \quad t > 0.$$

2. Resonance results

The principal tool is to reduce the multi-dimensional problem $(*)$, (B) to a one-dimensional problem. First we need two lemmas.

LEMMA 1. *If u is a solution of the problem $(*)$, (B), then $U(t)$ defined by (1) satisfies*

(2) $$U''(t) + \left(\sigma + \rho(\pi/L)^4\right)U'(t) + \left(\alpha(\pi/L)^4 + \beta(\pi/L)^2\right)U(t)$$

$$= F(t) + G(t),$$

where

$$\begin{aligned} F(t) &= \frac{1}{L}\int_0^L f(x,t)\sin{(\pi/L)x}\ dx, \\ G(t) &= L^{-1}\left(\alpha(\pi/L)^3 + \beta(\pi/L)\right)(g(t)+h(t)) \\ &\quad +\rho L^{-1}(\pi/L)^3\,(g'(t)+h'(t)) \\ &\quad -\alpha L^{-1}(\pi/L)\left(\tilde{g}(t)+\tilde{h}(t)\right) - \rho L^{-1}(\pi/L)\left(\tilde{g}'(t)+\tilde{h}'(t)\right). \end{aligned}$$

PROOF. Multiplying $(*)$ by $\psi = L^{-1}\sin{(\pi/L)x}$ and integrating over $[0, L]$, we have

$$U''(t) + \alpha\int_0^L \frac{\partial^4 u}{\partial x^4}\psi\,dx - \beta\int_0^L \frac{\partial^2 u}{\partial x^2}\psi\,dx + \sigma U'(t) + \rho\int_0^L \frac{\partial^5 u}{\partial x^4 \partial t}\psi\,dx = F(t). \tag{3}$$

Integration by parts yields

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\psi\,dx = L^{-1}(\pi/L)(g(t)+h(t)) - (\pi/L)^2\int_0^L u\psi\,dx, \tag{4}$$

$$\begin{aligned} &\int_0^L \frac{\partial^4 u}{\partial x^4}\psi\,dx \\ &= L^{-1}(\pi/L)\left(\tilde{g}(t)+\tilde{h}(t)\right) - L^{-1}(\pi/L)^3(g(t)+h(t)) \\ &\quad + (\pi/L)^4\int_0^L u\psi\,dx. \end{aligned} \tag{5}$$

We observe, using (5), that

$$\begin{aligned} &\int_0^L \frac{\partial^5 u}{\partial x^4 \partial t}\psi\,dx \\ &= \frac{d}{dt}\int_0^L \frac{\partial^4 u}{\partial x^4}\psi\,dx \\ &= L^{-1}(\pi/L)\left(\tilde{g}'(t)+\tilde{h}'(t)\right) - L^{-1}(\pi/L)^3\,(g'(t)+h'(t)) \\ &\quad + (\pi/L)^4\frac{d}{dt}\int_0^L u\psi\,dx. \end{aligned} \tag{6}$$

Combining (3)–(6), we obtain the desired differential equation (2).

LEMMA 2. *Let* $\alpha(\pi/L)^4 + \beta(\pi/L)^2 - 4^{-1}\left(\sigma + \rho(\pi/L)^4\right)^2 > 0$. *Then, the solution* $U(t)$ *of* (2) *can be written in the form*

$$\begin{aligned}(7)\qquad U(t) = {}& e^{-\mu t}\left(K_1 \sin\omega t + K_2 \cos\omega t\right)\\ & -\omega^{-1}\int_0^t (F(s)+G(s))e^{\mu(s-t)}\sin\omega(s-t)\,ds\end{aligned}$$

for some constants K_1 *and* K_2, *where*

$$\mu = 2^{-1}\left(\sigma + \rho(\pi/L)^4\right),$$

$$\omega = \left(\alpha(\pi/L)^4 + \beta(\pi/L)^2 - 4^{-1}\left(\sigma + \rho(\pi/L)^4\right)^2\right)^{1/2}.$$

PROOF. The differential equation (2) can be rewritten as

$$U''(t) + 2\mu U'(t) + \left(\mu^2 + \omega^2\right)U(t) = F(t) + G(t),$$

and therefore the solution $U(t)$ has the form (7).

Let P be a subset of $\mathbf{R}^1 \setminus \{\omega\}$ such that ω is an accumulation point of P. We consider the problem

$$(**)\quad \frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} - \beta\frac{\partial^2 u}{\partial x^2} + \sigma\frac{\partial u}{\partial t} + \rho\frac{\partial^5 u}{\partial x^4 \partial t} = f_p(x,t),\quad (x,t) \in (0,L)\times\mathbf{R}_+,$$

$$(\mathrm{B}_p)\quad u(0,t) = g_p(t),\ \frac{\partial^2 u}{\partial x^2}(0,t) = \tilde{g}_p(t),\ u(L,t) = h_p(t),\ \frac{\partial^2 u}{\partial x^2}(L,t) = \tilde{h}_p(t),\quad t > 0,$$

where $p \in P$ is a parameter, $f_p(x,t) \in \mathrm{C}\left(\overline{(0,\infty)\times\mathbf{R}_+}\right)$, and $g_p(t)$, $\tilde{g}_p(t)$, $h_p(t)$, $\tilde{h}_p(t)$ are functions of class $\mathrm{C}^1(\mathbf{R}_+)$. We define

$$\begin{aligned}F_p(t) &= \frac{1}{L}\int_0^L f_p(x,t)\sin(\pi/L)x\,dx,\\ G_p(t) &= L^{-1}\left(\alpha(\pi/L)^3 + \beta(\pi/L)\right)(g_p(t) + h_p(t))\\ &\quad + \rho L^{-1}(\pi/L)^3\left(g_p'(t) + h_p'(t)\right)\\ &\quad -\alpha L^{-1}(\pi/L)\left(\tilde{g}_p(t) + \tilde{h}_p(t)\right)\\ &\quad - \rho L^{-1}(\pi/L)\left(\tilde{g}_p'(t) + \tilde{h}_p'(t)\right).\end{aligned}$$

THEOREM 1. *Let* $\alpha(\pi/L)^4+\beta(\pi/L)^2-4^{-1}\left(\sigma+\rho(\pi/L)^4\right)^2>0$, $\mu>0$, *and let* $F_p(t)+G_p(t)=K\sin pt$, *where* K *is a nonzero constant. If* u_p *is a solution of the problem* $(**)$, (B_p), *then we have*

$$(8)\quad \max_{0\le x\le L}|u_p(x,t)|\ge \frac{|K|}{\left((p^2-\mu^2-\omega^2)^2+4p^2\mu^2\right)^{1/2}}|\sin(pt+\theta(p))| - \left(|\widetilde{K}_1(p)|+|\widetilde{K}_2(p)|\right)e^{-\mu t},$$

for some constants $\widetilde{K}_1(p)$ *and* $\widetilde{K}_2(p)$, *where*

$$\theta(p)=\operatorname{sgn}\left(p^2-\mu^2-\omega^2\right)\tan^{-1}\left(2p\mu/|p^2-\mu^2-\omega^2|\right).$$

Moreover, the amplitude $|K|\left((p^2-\mu^2-\omega^2)^2+4p^2\mu^2\right)^{-1/2}$ *has its maximum* $|K|(2\omega\mu)^{-1}$ *at* $p=p_1=(\omega^2-\mu^2)^{1/2}$.

PROOF. Lemma 2 implies that

$$(9)\quad U_p(t)=e^{-\mu t}\left(K_1(p)\sin\omega t+K_2(p)\cos\omega t\right) -\omega^{-1}K\int_0^t e^{\mu(s-t)}\sin\omega(s-t)\,\sin ps\,ds$$

for some constants $K_1(p)$ and $K_2(p)$, where

$$U_p(t)=\int_0^L u_p(x,t)\psi(x)\,dx.$$

We easily obtain

$$(10)\quad \int_0^t e^{\mu(s-t)}\sin\omega(s-t)\,\sin ps\,ds = -2^{-1}\int_0^t e^{\mu(s-t)}\left(\cos(ps+\omega(s-t))-\cos(ps-\omega(s-t))\right)ds.$$

Integration by parts yields

$$(11)\quad \int_0^t e^{\mu(s-t)}\cos(ps\pm\omega(s-t))\,ds = \mu\left(\mu^2+(p\pm\omega)^2\right)^{-1}\left(\cos pt-e^{-\mu t}\cos\omega t\right) + (p\pm\omega)\left(\mu^2+(p\pm\omega)^2\right)^{-1}\left(\sin pt\pm e^{-\mu t}\sin\omega t\right).$$

Combining (10) with (11), we obtain

$$
\begin{aligned}
(12)\quad & \int_0^t e^{\mu(s-t)} \sin\omega(s-t)\ \sin ps\ ds \\
= \ & \frac{1}{2}\Bigg[\left(\mu\left(\mu^2+(p-\omega)^2\right)^{-1} - \mu\left(\mu^2+(p+\omega)^2\right)^{-1}\right)\cos pt \\
& + \left((p-\omega)\left(\mu^2+(p-\omega)^2\right)^{-1} - (p+\omega)\left(\mu^2+(p+\omega)^2\right)^{-1}\right) \\
& \times \sin pt \\
& + \left(\mu\left(\mu^2+(p+\omega)^2\right)^{-1} - \mu\left(\mu^2+(p-\omega)^2\right)^{-1}\right) e^{-\mu t}\cos\omega t \\
& - \left((p+\omega)\left(\mu^2+(p+\omega)^2\right)^{-1} + (p-\omega)\left(\mu^2+(p-\omega)^2\right)^{-1}\right) \\
& \times e^{-\mu t}\sin\omega t\Bigg] \\
= \ & \left(\mu^2+(p-\omega)^2\right)^{-1}\left(\mu^2+(p+\omega)^2\right)^{-1} \\
& \times \Big[2p\omega\mu\ \cos pt + \omega\left(p^2-\omega^2-\mu^2\right)\sin pt - 2p\omega\mu\, e^{-\mu t}\cos\omega t \\
& -p\left(\mu^2+p^2-\omega^2\right)e^{-\mu t}\sin\omega t\Big] \\
= \ & \omega\left(\mu^2+(p-\omega)^2\right)^{-1}\left(\mu^2+(p+\omega)^2\right)^{-1} \\
& \times\sqrt{\left(p^2-\omega^2-\mu^2\right)^2+4p^2\mu^2}\,\operatorname{sgn}\left(p^2-\omega^2-\mu^2\right)\,\sin(pt+\theta(p)) \\
& - \left(\mu^2+(p-\omega)^2\right)^{-1}\left(\mu^2+(p+\omega)^2\right)^{-1} 2p\omega\mu\, e^{-\mu t}\cos\omega t \\
& - \left(\mu^2+(p-\omega)^2\right)^{-1}\left(\mu^2+(p+\omega)^2\right)^{-1} p\left(\mu^2+p^2-\omega^2\right) \\
& \times e^{-\mu t}\sin\omega t.
\end{aligned}
$$

Substituting (12) into (9) yields

$$
\begin{aligned}
U_p(t) \ = \ & e^{-\mu t}\left(\widetilde{K}_1(p)\sin\omega t + \widetilde{K}_2(p)\cos\omega t\right) \\
& -K\left(\left(p^2-\omega^2-\mu^2\right)^2+4p^2\mu^2\right)^{-1/2}\operatorname{sgn}\left(p^2-\omega^2-\mu^2\right) \\
& \times\sin\left(pt+\theta(p)\right),
\end{aligned}
$$

where

$$
\widetilde{K}_1(p) \ = \ K_1(p) + \omega^{-1}Kp\left(\mu^2+p^2-\omega^2\right)\left(\mu^2+(p-\omega)^2\right)^{-1}\times
$$

$$\begin{aligned}
&\times\left(\mu^2+(p+\omega)^2\right)^{-1}, \\
\widetilde{K}_2(p) \quad &= \quad K_2(p)+2Kp\mu\left(\mu^2+(p-\omega)^2\right)^{-1}\left(\mu^2+(p+\omega)^2\right)^{-1},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\max_{0\le x\le L}|u_p(x,t)| \ge |U_p(t)| \\
&\quad \ge |K|\left(\left(p^2-\omega^2-\mu^2\right)^2+4p^2\mu^2\right)^{-1/2}|\sin(pt+\theta(p))| \\
&\quad - \left(|\widetilde{K}_1(p)|+|\widetilde{K}_2(p)|\right)e^{-\mu t},
\end{aligned}$$

which is the desired estimate (8). In view of the fact that the function $h(p) = \left((p^2-\omega^2-\mu^2)^2+4p^2\mu^2\right)^{1/2}$ has its minimum $2\omega\mu$ at $p = p_1 = (\omega^2-\mu^2)^{1/2}$, we conclude that the second statement holds.

THEOREM 2. *Assume that* $\mu = 0$ *and* $\alpha(\pi/L)^4+\beta(\pi/L)^2 > 0$. *Let* $F_p(t)+G_p(t) = K\cos(pt+t_0)$, *where* K *is a nonzero constant and* t_0 *is some number, and let* u_p *be a solution of the problem* (**), (B_p). *Then, we see that:*

(i) $\limsup_{p\to\omega}\max_{0\le x\le L}|u_p(x,t)| = \infty$ *for all* $t \in T_1\left[(u_p)_{p\in P}\right]$,

(ii) *there is a constant* C *such that*

$$\begin{aligned}
&\limsup_{p\to\omega}\max_{0\le x\le L}|u_p(x,t)| \\
&\quad \ge |K|(2\omega)^{-1}|\,t\sin(\omega t+t_0)-\omega^{-1}\sin\omega t\,\sin t_0|-C
\end{aligned}$$

for all $t \in T_2\left[(u_p)_{p\in P}\right]$, *where*

$$\begin{aligned}
T_1\left[(u_p)_{p\in P}\right] \quad &\equiv \quad \left\{t\in[0,\infty)\,;\ \limsup_{p\to\omega}\max_{0\le x\le L}|u_p(x,t)| = \infty\right\}, \\
T_2\left[(u_p)_{p\in P}\right] \quad &\equiv \quad \left\{t\in[0,\infty)\,;\ \limsup_{p\to\omega}\max_{0\le x\le L}|u_p(x,t)| < \infty\right\}.
\end{aligned}$$

PROOF. It is obvious that the first statement (i) holds. Let $t \in T_2\left[(u_p)_{p\in P}\right]$, i.e. $\limsup_{p\to\omega}\max_{0\le x\le L}|u_p(x,t)| < \infty$. Proceeding as in the proof of Lemma 1, we obtain

$$(13) \qquad U_p(t) = K_1(p)\sin\omega t + K_2(p)\cos\omega t - \omega^{-1}K\int_0^t \cos(ps+t_0)\ \sin\omega(s-t)\,ds$$

for some constants $K_1(p)$ and $K_2(p)$. From (13) it follows that

$$\begin{aligned} & |K_1(p)\sin\omega t + K_2(p)\cos\omega t| \\ \le\ & \max_{0\le x\le L}|u_p(x,t)| \ + \omega^{-1}|K|\left|\int_0^t \cos(ps+t_0)\ \sin\omega(s-t)\,ds\right|. \end{aligned}$$

It is easily verified that

$$\begin{aligned} & \lim_{p\to\omega}\int_0^t \cos(ps+t_0)\ \sin\omega(s-t)\,ds \\ =\ & -2^{-1}t\,\sin(\omega t+t_0) + (2\omega)^{-1}\sin t_0\ \sin\omega t. \end{aligned}$$

Hence,

$$\limsup_{p\to\omega}|K_1(p)\sin\omega t + K_2(p)\cos\omega t| < \infty$$

for each $t \in T_2\left[(u_p)_{p\in P}\right]$. It holds that

$$|K_1(p)\sin\omega t + K_2(p)\cos\omega t| = \left(K_1(p)^2 + K_2(p)^2\right)^{1/2}|\sin(\omega t+\theta_p)|$$

for some constant θ_p. Now, we observe that either

$$\limsup_{p\to\omega}\left(K_1(p)^2 + K_2(p)^2\right)^{1/2} < \infty,$$

or else there exists a sequence $(p_k)_{k\in\mathbf{N}}$ such that

$$\left(K_1(p_k)^2 + K_2(p_k)^2\right)^{1/2} \to \infty \quad (k\to\infty).$$

In the former case, we see that

$$(14) \qquad \sup_{t\in T_2\left[(u_p)_{p\in P}\right]}\ \limsup_{p\to\omega}|K_1(p)\sin\omega t + K_2(p)\cos\omega t| < \infty.$$

In the latter case, we have $\sin(\omega t+\theta_{p_k}) \to 0$ $(k\to\infty)$ for all $t \in T_2\left[(u_p)_{p\in P}\right]$, and hence $\lim_{k\to\infty}(\omega t+\theta_{p_k}) \in \pi\mathbf{Z}$. Therefore, $t_2 - t_1 \in$

$\frac{\pi}{\omega}\mathbf{Z}$ for all $t_1, t_2 \in T_2\left[(u_p)_{p\in P}\right]$, i.e. $t_2 = t_1 + \frac{\pi}{\omega}m$ for some $m \in \mathbf{Z}$. Then we have

$$\begin{aligned} & \limsup_{p\to\omega} |K_1(p)\sin\omega t_2 + K_2(p)\cos\omega t_2| \\ = & \limsup_{p\to\omega} |(-1)^m (K_1(p)\sin\omega t_1 + K_2(p)\cos\omega t_1)| \\ = & \limsup_{p\to\omega} |K_1(p)\sin\omega t_1 + K_2(p)\cos\omega t_1| . \end{aligned}$$

Hence, $\limsup_{p\to\omega} |K_1(p)\sin\omega t + K_2(p)\cos\omega t|$ is independent of $t \in T_2\left[(u_p)_{p\in P}\right]$. Hence, (14) holds also in this case. By (13) we obtain

$$\begin{aligned} \max_{0\le x\le L} |u_p(x,t)| \ \ge \ & \omega^{-1}|K| \left|\int_0^t \cos(ps+t_0)\ \sin\omega(s-t)\,ds\right| \\ & -|K_1(p)\sin\omega t + K_2(p)\cos\omega t| \end{aligned}$$

and therefore

$$\begin{aligned} & \limsup_{p\to\omega} \max_{0\le x\le L} |u_p(x,t)| \\ \ge \ & (2\omega)^{-1}|K| \left| t\ \sin(\omega t + t_0) - \omega^{-1}\sin t_0\ \sin\omega t\right| \\ & - \limsup_{p\to\omega} |K_1(p)\sin\omega t + K_2(p)\cos\omega t|. \end{aligned}$$

In view of (14), we conclude that the statement (ii) holds. The proof is complete.

By choosing $t_0 = -\frac{\pi}{2}$ in Theorem 2, we obtain the following corollary.

COROLLARY. *Assume that $\mu = 0$. Let $F_p(t)+G_p(t) = K\sin pt$, where K is a nonzero constant, and let u_p be a solution of the problem* (**), (B_p). *Then we see that:*

(i) $\limsup_{p\to\omega} \max_{0\le x\le L} |u_p(x,t)| = \infty$ *for all* $t \in T_1\left[(u_p)_{p\in P}\right]$,

(ii) *there exists a constant C such that*

$$\limsup_{p\to\omega} \max_{0\le x\le L} |u_p(x,t)| \ge |K|(2\omega)^{-1}t\ |\cos\omega t| - C$$

for all $t \in T_2\left[(u_p)_{p\in P}\right]$.

3. Examples

We give some examples which illustrate Corollary.

EXAMPLE 1. We consider the problem

$$\frac{\partial^2 u}{\partial t^2}+\frac{1}{2}\frac{\partial^4 u}{\partial x^4}-\frac{1}{2}\left(\frac{\pi}{L}\right)^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial t}-\left(\frac{L}{\pi}\right)^4\frac{\partial^5 u}{\partial x^4\partial t} = 2K\sin pt\ \sin(\pi/L)x,\quad (x,t)\in(0,L)\times\mathbf{R}_+, \tag{15}$$

$$u(0,t)=\frac{\partial^2 u}{\partial x^2}(0,t)=u(L,t)=\frac{\partial^2 u}{\partial x^2}(L,t)=0,\quad t>0, \tag{16}$$

where K is a nonzero constant. Here $\alpha=2^{-1}$, $\beta=2^{-1}(\pi/L)^2$, $\sigma=1$, $\rho=-(L/\pi)^4$, and $f_p(x,t)=2K\sin pt\ \sin(\pi/L)x$. It is easy to see that $\mu=0$, $\omega=(\pi/L)^2$ and $F_p(t)+G_p(t)=K\sin pt$. There is a solution

$$\begin{aligned} u_p(x,t) &= \omega^{-1}2K\left((p+\omega)^{-1}\sin pt-p(p+\omega)^{-1}t\,\frac{\sin pt-\sin\omega t}{pt-\omega t}\right)\\ &\quad\times\sin(\pi/L)x \end{aligned}$$

of the problem (15), (16) if $p\neq\omega$. Since

$$\begin{aligned} &\max_{0\le x\le L}|u_p(x,t)|\\ &= \omega^{-1}2|K|\left|(p+\omega)^{-1}\sin pt-p(p+\omega)^{-1}t\,\frac{\sin pt-\sin\omega t}{pt-\omega t}\right|, \end{aligned}$$

it can be shown that

$$\begin{aligned} &\limsup_{p\to\omega}\max_{0\le x\le L}|u_p(x,t)|\\ &= \omega^{-1}2|K|\left|(2\omega)^{-1}\sin\omega t-2^{-1}t\,\cos\omega t\right|\\ &\ge \omega^{-1}|K|\left|(2\omega)^{-1}\sin\omega t-2^{-1}t\,\cos\omega t\right|\\ &\ge |K|(2\omega)^{-1}t\,|\cos\omega t|-|K|\left(2\omega^2\right)^{-1} \end{aligned}$$

for all $t\in T_2\left[(u_p)_{p\in P}\right]\equiv[0,\infty)$. In this case, we find that $T_1\left[(u_p)_{p\in P}\right]=\emptyset$ and the second statement (ii) of Corollary holds with $C=|K|\,(2\omega^2)^{-1}$.

EXAMPLE 2. We consider the problem

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in (0,L) \times \mathbf{R}_+, \tag{17}$$

$$\begin{aligned} u(0,t) &= \frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad u(L,t) = KA^{-1} \sin pt, \\ \frac{\partial^2 u}{\partial x^2}&(L,t) = -(\pi/(2L))^2 KA^{-1} \sin pt, \quad t > 0, \end{aligned} \tag{18}$$

where K is a nonzero constant, $p^2 = \alpha(\pi/(2L))^4 + \beta(\pi/(2L))^2$, $\omega = (\alpha(\pi/L)^4 + \beta(\pi/L)^2)^{1/2}$ and

$$A = L^{-1}\left(\alpha(\pi/L)^3 + \beta(\pi/L)\right) + \alpha L^{-1}(\pi/L)(\pi/(2L))^2.$$

Here $\sigma = \rho = 0$ and $f_p(x,t) \equiv 0$. It is easily seen that $F_p(t) \equiv 0$ and $G_p(t) = K \sin pt$. It can be shown that $A = A(p) = 4(3\pi)^{-1}(\omega^2 - p^2)$ and

$$u_p(x,t) = KA^{-1} \sin pt \ \sin(\pi/(2L))x$$

is a solution of the problem (17), (18) if $p \neq \omega$. Since

$$\max_{0 \le x \le L} |u_p(x,t)| = |K| \left| \frac{\sin pt}{A(p)} \right|,$$

we observe that

$$\begin{aligned} &\limsup_{p \to \omega} \max_{0 \le x \le L} |u_p(x,t)| \\ = \ &\limsup_{p \to \omega} |K| 4^{-1}(3\pi)|p + \omega|^{-1} \left| \frac{\sin pt}{p - \omega} \right| \\ = \ &|K|\,(3\pi/4)\,(2\omega)^{-1} t\,|\cos \omega t| \\ \geq \ &|K|\,(2\omega)^{-1} t\,|\cos \omega t| \end{aligned}$$

for $t \in T_2\left[(u_p)_{p \in P}\right] \equiv \{\omega^{-1} n\pi;\ n = 0, 1, 2, \ldots\}$ and

$$\limsup_{p \to \omega} \max_{0 \le x \le L} |u_p(x,t)| = \infty$$

for $t \in T_1\left[(u_p)_{p \in P}\right] \equiv [0, \infty) \setminus T_2\left[(u_p)_{p \in P}\right]$. In this case, the second statement (ii) of Corollary holds with $C = 0$.

EXAMPLE 3. We consider the problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in (0,L) \times \mathbf{R}_+, \tag{19}$$

$$u(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad u(L,t) = A(p)^{-1} \sin pt, \tag{20}$$

$$\frac{\partial^2 u}{\partial x^2}(L,t) = -A(p)^{-1}\left(\left(1+p^2\right)^{1/2} - 1\right) \sin pt, \quad t > 0,$$

where $A(p) = L^{-2}\pi\left((\pi/L)^2 + (1+p^2)^{1/2} + 1\right)$. Here $\alpha = 1$, $\beta = 2$, $\sigma = \rho = 0$ and $f_p(x,t) \equiv 0$. It is readily seen that $F_p(t) \equiv 0$, $G_p(t) = \sin pt$ and $\omega = (\pi/L)\left((\pi/L)^2 + 2\right)^{1/2}$. If $p \neq \omega$, we find that

$$u_p(x,t) = A(p)^{-1}(\sin\theta(p)L)^{-1}\sin\theta(p)x \ \sin pt$$

is a solution of the problem (19), (20), where

$$\theta(p) = \left(\left(1+p^2\right)^{1/2} - 1\right)^{1/2}.$$

Since

$$\lim_{p\to\omega}\theta(p) = \pi/L,$$

$$\lim_{p\to\omega}A(p) = 2\pi L^{-2}\left((\pi/L)^2 + 1\right),$$

$$\lim_{p\to\omega} \sin\ \theta(p)L = \sin\pi = 0$$

and

$$\lim_{p\to\omega}\sin pt = \sin\omega t \neq 0$$

for $t \in T_1\left[(u_p)_{p\in P}\right] \equiv [0,\infty)\setminus\{\omega^{-1}n\pi;\ n = 0,1,2,\ldots\}$, we easily see that

$$\begin{aligned}&\limsup_{p\to\omega}\max_{0\le x\le L}|u_p(x,t)| \\ = \ &\limsup_{p\to\omega}\ |A(p)|^{-1}\left|\frac{\sin pt}{\sin\theta(p)L}\right| = \infty\end{aligned}$$

for $t \in T_1\left[(u_p)_{p\in P}\right]$. In view of the fact that

$$\lim_{p\to\omega}\frac{\sin pt}{\sin\theta(p)L} = (-1)^{n+1}2\pi L^{-2}\left((\pi/L)^2 + 1\right)\omega^{-1}t$$

for $t \in T_2\left[(u_p)_{p\in P}\right] \equiv \{\omega^{-1}n\pi;\ n = 0, 1, 2, \ldots\}$, we conclude that

$$\begin{aligned}
&\limsup_{p\to\omega} \max_{0\le x\le L} |u_p(x,t)| \\
=\ &\limsup_{p\to\omega} |A(p)|^{-1} \left| \frac{\sin pt}{\sin\theta(p)L} \right| \\
=\ &\omega^{-1}t \ge (2\omega)^{-1}t
\end{aligned}$$

for $t \in T_2\left[(u_p)_{p\in P}\right]$. In this case, the statement (ii) of Corollary is satisfied with $C = 0$.

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Department of Mathematics
Faculty of Science
Toyama University
Toyama 930
JAPAN

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